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Towards the graviton from spinfoams: the 3d toy model

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Abstract: Recently, a proposal has appeared for the extraction of the 2-point function of linearised quantum gravity, within the spinfoam formalism. This relies on the use of a boundary state, which introduces a semi–classical flat geometry on the boundary. In this paper, we investigate this proposal considering a toy model in the (Riemannian) 3d case, where the semi–classical limit is better understood. We show that in this limit the propagation kernel of the model is the one for the harmonic oscillator. This is at the origin of the expected $1/\ell$ behaviour of the 2-point function. Furthermore, we numerically study the short scales regime, where deviations from this behaviour occur.

Keywords: Lattice Models of Gravity, Models of Quantum Gravity.

Contents

1. Introduction

The spinfoam formalism [1] is a candidate covariant approach to a non–perturbative quantisation of General Relativity (GR). At present, it seems to provide a consistent background independent theory at the Planck scale, where it describes a (discrete) quantum geometry. However, the large scale behaviour is less understood. In particular, the formalism lacks a well–defined procedure to study the semi–classical limit, define particle scattering amplitudes and reproduce low–energy physics. Consider the pure gravity case: in the low–energy limit, we would expect to recover the conventional perturbative expansion described in terms of gravitons. The importance of obtaining this would be two–fold: on the one hand, it would provide fundamental ground for the emergence of General Relativity in the large scale limit of spinfoams, thus proving it to be a sensible theory of quantum gravity. On the other hand, it would open the way to the exploration of the corrections to the propagation of gravitons due to the microscopical quantum geometry, where new physics is potentially expected.

There is a simple heuristic reason that shows the difficulty to recover conventional quantum field theory (QFT) from background independent approaches: the arguments of the n-point functions used in QFT are spacetime point coordinates, which are not defined in the absence of a background. Recently, a possible way out of this problem has been suggested in [2]. The key idea is to construct n-point functions by means of the propagation kernel, which provides an amplitude for the fields assigned on a boundary of spacetime. Since the value of the gravitational field provides the boundary with a metric,

n-point functions can be defined with respect to this metric. In the deep quantum gravity regime, one is willing to consider arbitrary boundaries [3, 4]. However, to recover the behaviour of linearised quantum gravity, it seems as a first step sufficient to simply consider the usual setting of QFT, when the boundary is given by two spacelike hyperplanes, and the boundary values of the gravitational field are small perturbations over flat space, h'_{ab} and h''_{ab}, which are asymptotically vanishing. In this context, the propagation kernel $W[h', h'', T]$ for linearised GR was considered in [5, 6], and explicitly derived in the temporal gauge in [7]. The time T entering the kernel is the asymptotic proper time between the two hyperplanes. Indeed, it is a simple exercise to recover for instance the 2-point function $W_{abcd}(x, y)$ by sandwiching $W[h', h'', T]$ with two one–particle states $\Psi_1[h']_{ab}(x)$, $\Psi_1[h'']_{cd}(y)$, and integrating over the boundary values h'_{ab} and h''_{ab}:

$$W_{abcd}(\vec{x}, \vec{y}; T) = \frac{1}{\mathcal{N}} \int \mathcal{D}h' \ \mathcal{D}h'' \ W[h', h'', T] \ \Psi_0[h'] \ h'_{ab}(0, \vec{x}) \ \Psi_0[h''] \ h''_{cd}(T, \vec{y}), \quad (1.1)$$

where the normalisation constant $\mathcal{N}$ is given by the functional integral without the field insertions.[1] Notice that both the kernel W and the vacuum state Ψ_0 (encoded in the kernel, see [7]) are gauge invariant expressions; however, the field insertions h'_{ab} and h''_{cd} are not gauge invariant, thus the evaluation of the expression above requires an additional spatial gauge–fixing. Choosing for instance the Coulomb–like gauge $\partial_a h_{ab} = 0$, we have

$$W_{abcd}(\vec{x}, \vec{y}; T) = -\ell_{\rm P} \int \frac{d^{n-1}p}{(2\pi)^{n-1}} \ \frac{e^{-ip(x-y)}}{2\omega_p} \ \Big(D_{ac} D_{bd} + D_{ad} D_{bc} - \frac{2}{n-2} D_{ab} D_{cd} \Big), \quad (1.2)$$

where n is the dimension of spacetime, and $D_{ab} = \delta_{ab} - \frac{p_a p_b}{p^2}$ is the transverse projector. Different gauge choices result in different numerical factors in the tensorial structure.

Following [2], the 2-point function of linearised quantum gravity (1.1) can be translated into the spinfoam formalism using the expression

$$W_{\mu\nu\rho\sigma}(x, y) = \frac{1}{\mathcal{N}} \sum_s \ W[s] \ \Psi_0[s] \ \langle s | h_{\mu\nu}(x) | s \rangle \ \langle s | h_{\rho\sigma}(y) | s \rangle. \quad (1.3)$$

Here the sum is over spin networks s; the propagation kernel $W[s]$ is provided by the explicit spinfoam model chosen, and it includes a sum over all spinfoams σ compatible with s. The boundary state $\Psi_0[s]$ is a weighted functional of spin networks peaked around those reproducing flat space in the semi–classical limit.

This approach has been considered in [8] for a proposal to extract the graviton from the (Riemannian) 4d Barrett–Crane model. The results there described are promising; indeed, considering a single component of the 2-point function, one does recover the correct $1/\ell^2$ dependence on the distance. However, there are many issues that still needs to be clarified. In particular, the role played by the Gaussian boundary state used to introduce the flat geometry on the boundary, and the way this geometry determines the distance dependence.

Remarkably, these issues are present in a very similar way also in the 3d case. The 3d case has the advantage that the semi–classical limit is better understood, and there are

[1] Strictly speaking, this is only the advanced Green function $\Delta_+(x, y)$. The Feynman propagator is obtained by taking $\theta(T)\Delta_+(x, y) + \theta(-T)\Delta_-(x, y)$.

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no degenerate configurations plaguing the limit. In this paper, we propose to apply the strategy of [8] in 3d, in the hope to clarify some of the issues. As it is well known, 3d GR has no local degrees of freedom. Nonetheless, the 2-point function of the linearised quantum theory is a well defined quantity that can be evaluated once a gauge–fixing is chosen. For instance, it is given in the Coulomb gauge by (1.2) with $n = 3$; it shows the usual $1/p^2$ behaviour, which in 3d means a dependence $1/\ell$ over the spacetime distance. However, this quantity is a pure gauge, thus the quantum theory does not have a propagating graviton [9].

Notice also that in 3d the Newton constant G has inverse mass dimensions (in units $c = 1$); we define the 3d Planck length as $\ell_{\rm P} = 16\pi\hbar G$.

2. The toy model

The toy model we consider here is a tetrahedron with dynamics described by the Regge action, whose fundamental variables are the edge lengths ℓ_e. Since we are dealing with a single tetrahedron, all edges are boundary edges, and the action consists only of the boundary term, namely it coincides with the Hamilton function of the system:

$$S_{\rm R}[\ell_e] = \frac{1}{16\pi G} \sum_e \ell_e \, \theta_e(\ell_e). \tag{2.1}$$

Here the θ_e are the dihedral angles of the tetrahedron, namely the angles between the outward normals to the triangles. They represent a discrete version of the extrinsic curvature [10]. Furthermore, they satisfy the non–trivial relation

$$16\pi G \, \frac{\partial S_{\rm R}}{\partial \ell_e} = \theta_e. \tag{2.2}$$

In this discrete setting, assigning the six edge lengths is equivalent to the assignment of the boundary gravitational field. In particular, let us consider the case when two opposite edges of the tetrahedron have lengths a and b, and the remaining four edges have all the same length, say c. Following [11], we know that the Hamilton function $S[a, b, c]$ given by (2.1) defines a simple relativistic system, of which a, b and c are partial observables [12]. That is, they include both the independent (time) variable, and the dependent (dynamical) variables, and the dynamics provides a relation amongst them. In the following, we choose c as the time variable. The classical dynamics of the system is described using the Hamilton function to read the evolution of b in a time c, given the initial value a.

The quantum dynamics, on the other hand, is described by the Ponzano–Regge (PR) model [13]. In the model, the lengths are promoted to operators $\ell_{\rm P}^2 X_e^I X_e^I$, whose spectrum is given by $\ell_e^2 = \ell_{\rm P}^2 C^2(j_e)$, where the half–integer j labels SU(2) irreducible representations (irreps), and $C^2(j) = j(j+1) + \frac{1}{4}$ is the Casimir operator.[2] In the model, each tetrahedron has an amplitude given by Wigner's $\{6j\}$ symbol for the recoupling theory of SU(2). Under

[2]Conventionally, the SU(2) Casimir is taken to be simply $j(j+1)$. However, it is defined up to an additive constant, and the value 1/4 chosen here allows a matching with the original PR ansatz, $\ell_e = \ell_{\rm P}(j_e + \frac{1}{2})$.

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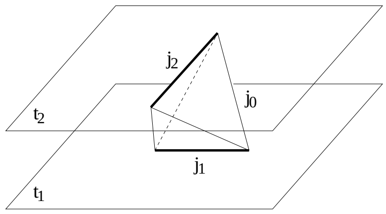

Figure 1: The dynamical tetrahedron as evolution between two hyperplanes. The labels give the physical lengths as $a = \ell_{\rm P} C(j_1)$, $b = \ell_{\rm P} C(j_2)$, $c = \ell_{\rm P} C(j_0)$, and $T = t_2 - t_1 = c/\sqrt{2}$.

the identifications $a = \ell_{\rm P} C(j_1)$, $b = \ell_{\rm P} C(j_2)$, $c = \ell_{\rm P} C(j_0)$, the propagation kernel for our toy model is

$$W[j_1, j_2, j_0] = \prod_e (2j_e + 1) \begin{Bmatrix} j_1 & j_0 & j_0 \\ j_2 & j_0 & j_0 \end{Bmatrix}. \tag{2.3}$$

It represents the amplitude for the (eigen)values of the boundary gravitational field. Notice that we can assume to have measured the quantity j_0, and then interpret $W[j_1, j_2, j_0]$ as the transition amplitude between j_1 and j_2 in a time j_0.

The key property of the PR model which we will need in the following is that in the large j regime we have [13–15]

$$\lim_{j \mapsto \infty} W[j_1, j_2, j_0] \sim \prod_e (2j_e + 1) \sqrt{\frac{2}{3\pi V(j_e)}} \cos\left(\sum_e (j_e + \frac{1}{2}) \theta_e(j_e) + \frac{\pi}{4} \right), \tag{2.4}$$

where the argument of the cosine can be recognised to be ($1/\hbar$ times) the Regge action (2.1), with $\ell_e = \ell_{\rm P} C(j_e)$ (the constant factor $\frac{\pi}{4}$ appearing in (2.4) clearly does not affect the classical dynamics). Usually, we would expect the semi–classical limit to be the exponential of the classical action. However, the presence of the two exponentials giving rise to the cosine can be understood as follows. The sign in front of the action is determined by the orientation of the manifold; it is reversed if we invert the orientation of M. However, the classical theory does not distinguish between the two orientation, thus the path integral sums over both. We will see below that only one contributes to the propagator. Therefore, the model admits a clear semi–classical limit, obtained by taking $j \mapsto \infty$, $\ell_{\rm P} \mapsto 0$, such that the product $\ell_{\rm P}\, j$ is finite.

3. The boundary state

To make a link between this toy model and the 2-point function of GR, we consider two hyperplanes in 3d Euclidean spacetime, and we embed the tetrahedron as in figure 1. Namely, we assume that we have measured the time interval $T := t_2 - t_1$, and we compute the correlator between perturbations of the length of the bottom edge, say ℓ_a, and perturbations of the top edge, ℓ_b. Because the two edges are embedded in flat space, we can assume the coordinate vector ℓ_a^μ to be along the x^1 direction and ℓ_b^μ along the x^2 direction, as in Fig.1. With this setting, we are naturally led to consider the projection $\ell_a^\mu(x) \ell_a^\nu(x) \ell_b^\rho(y) \ell_b^\sigma(y) W_{\mu\nu\rho\sigma}(x, y)$ of (1.3), where x and y are the mid–points respectively of

ℓ_a and ℓ_b; therefore we restrict our attention to the following quantity,

$$\ell_a^2\,\ell_b^2\,W_{1122}(T) := \ell_a^\mu(x)\ell_a^\nu(x)\ell_b^\rho(y)\ell_b^\sigma(y)W_{\mu\nu\rho\sigma}(x,y) = \ell_a^2\,\ell_b^2\,f_\lambda\ell_{\rm P}\int\frac{d^2p}{(2\pi)^2}\frac{e^{i\omega_p T}}{2\omega_p}, \tag{3.1}$$

where f_λ is a numerical factor depending on the choice of gauge (in (1.2), $f_\lambda = 2$).

In this "minimalistic" toy model, the boundary spin network s appearing in (1.3) is the boundary spin network of the tetrahedron. Since this is a tetrahedral graph with links in one–to–one correspondence with the edges of the tetrahedron, we can simply use j_e for the spin network labels. However, we stress that we are considering a situation where we have measured the "time" variable T, thus the labels for the edges in the bulk are fixed to the value j_0.[3] With a little trigonometry, we have $T = c/\sqrt{2}$. Therefore, the sum over s in (1.3) is truly simply a sum over j_1 and j_2, and (2.3) is the propagation kernel to be used,

$$\ell_a^2\,\ell_b^2\,W_{1122}(x,y) = \frac{1}{\mathcal{N}}\sum_{j_1,j_2} W[j_1,j_2,j_0]\;\Psi_0[j_1,j_2]\;\langle j_1|\ell_a^\mu\ell_a^\nu h_{\mu\nu}(x)|j_1\rangle\;\langle j_2|\ell_b^\rho\ell_b^\sigma h_{\rho\sigma}(y)|j_2\rangle. \tag{3.2}$$

The next ingredient entering (1.3) is the vacuum boundary state, $\Psi_0[j_1,j_2]$. Its role is to peak the boundary spin networks around those reproducing flat spacetime. The natural choice is to take a Gaussian around a background value for the variables j_1 and j_2. With respect to this, notice that any value of j_1 and j_2 is compatible with a flat tetrahedron. For simplicity, let us choose an equilateral tetrahedron: since the bulk edges are fixed to be j_0 by the time measurement, this means peaking the Gaussian around the value j_0 for both j_1 and j_2. Furthermore, a key observation in [8] is that the Gaussian should be peaked on the intrinsic *as well as* the extrinsic geometry of the boundary.[4] While the intrinsic classical geometry is represented by the edge lengths, the extrinsic geometry, on the other hand, must be fixed giving the boundary dihedral angles θ_e. For the equilateral tetrahedron, the value of all dihedral angles is $\vartheta = \arccos(-\frac{1}{3})$. Using j_0 and ϑ, we consider the tentative boundary state

$$\Psi_0[j_1,j_2] = \frac{1}{\mathcal{N}_0}\exp\Big\{-\frac{\alpha}{2}\sum_{i=1}^{2}(j_i-j_0)^2 + i\vartheta\sum_{i=1}^{2}(j_i+\frac{1}{2})\Big\}. \tag{3.3}$$

Here α is the width of the Gaussian, which we leave as a free parameter for the moment. However, let us notice the following. In order for (3.3) to represent a good vacuum boundary state, we require that in the semi–classical limit all relative uncertainties vanish, that is

$$\frac{\langle\Delta j\rangle}{\langle j\rangle} = \frac{1}{j_0\sqrt{\alpha}} \mapsto 0, \qquad \frac{\langle\Delta\theta\rangle}{\langle\theta\rangle} = \frac{\sqrt{\alpha}}{\vartheta} \mapsto 0. \tag{3.4}$$

[3] Consequently the conjugate variables, namely the bulk dihedral angles, are maximally spread. Notice that this is different from the setting of [8], where all labels are allowed to fluctuate. However, a link between the two settings can be made by considering in [8] a Gaussian with different widths for the bulk and the top and bottom edges, and then studying the limit case when the former goes to zero.

[4] This is analogous to a Gaussian representing a coherent state for a free particle in QM, which is picked on both the position and momentum.

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The semi–classical limit is obtained by taking $j_0 \mapsto \infty$. Since ϑ is not affected by this limit, the conditions above imply $\alpha \propto j_0^{-r}$, with $0 < r < 2$. In spite of its simplicity, we show below that this Gaussian is enough to characterize the usual properties of the propagator over flat spacetime.

4. The field insertions

The last ingredients entering (3.2) are the expectation values of the gravitational operator, such as $\langle j_1 | \ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) | j_1 \rangle$. A crucial point here concerns gauge invariance. Let us go back to (1.1). As discussed in the introduction, the field insertions are not gauge invariant quantities, therefore an additional gauge–fixing is required, in order for (1.1) to be well defined. Precisely in the same way, a gauge–fixing has to be chosen to give a well defined prescription to identify $\langle j_1 | \ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) | j_1 \rangle$ in the PR model, and thus to evaluate (3.2).

In particular, notice that we can choose in the continuum the gauge $\ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) = 0$; to be consistent with this choice, we have to set $\langle j_1 | \ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) | j_1 \rangle = 0$ in the PR model. Consequently, (3.2) is identically zero. The peculiarity of the 3d case is that this choice of gauge can be made simultaneously for all independent directions: the 3d graviton propagator is a pure gauge, and accordingly all the components can be set to zero. In the 4d case, on the other hand, the gauge choice $\ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) = 0$ cannot be made for all independent directions; in the end two components survive, the two degrees of freedom of a spin-2 massless particle. However, in the construction presented here we consider a single component of the graviton, thus we can not give a full proof of this peculiarity. To pursue this further, we need to extend the model to more tetrahedra, so that we can study the full tensorial structure. In fact, once we have the full tensorial structure, the counting of the number of physical degrees of freedom can be done using the Ward identities, which relate different diagrams through the symmetries of the theory. The relevant symmetry of 3d GR is the invariance under diffeomorphisms, which in the PR model is implemented by the so–called Pachner moves (see for instance [17]): these moves involve changing the number of tetrahedra in the triangulation. Therefore, in order to apply these moves and study the symmetries of the 2-point function, we have to extend the model to more tetrahedra, and we leave this issue open for further work.

To study the large scale behaviour emerging from (3.2), we now proceed choosing a Coulomb–like gauge in which $h_{\mu\nu}$ has components along the edge ℓ_a, so that its projections can be identified with non–trivial PR operators, such as $X_a^I X_a^I$, the length operator introduced above. More precisely, since $h_{\mu\nu}$ is a small perturbation around flat spacetime, we can write $\ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) = \ell_a^\mu \ell_a^\nu (g_{\mu\nu}(x) - \delta_{\mu\nu}) := \ell_{\mathrm{P}}^2 X_a^I X_a^I - \ell_{\mathrm{P}}^2 C^2(j_0)$, where $\ell_{\mathrm{P}}^2 C^2(j_0)$ is the background value of the length. Recall that $X_a^I X_a^I$ acts diagonally on an edge labeled by j, giving $C^2(j)$; therefore, in this gauge, the semi–classical limit $j_0 \mapsto \infty$ gives

$$\langle j_1 | \ell_a^\mu \ell_a^\nu h_{\mu\nu}(x) | j_1 \rangle = \ell_{\mathrm{P}}^2 [C^2(j_1) - C^2(j_0)] \sim 2 j_0 (j_1 - j_0) \ell_{\mathrm{P}}. \tag{4.1}$$

5. Graviton propagator

We are now ready to perform the evaluation of (3.2). To shorten our notation, let us define $\delta j_i := j_i - j_0$, $i = 1, 2$. Using the explicit expressions (2.4), (3.3) and (4.1), we have

$$\ell_a^2\, \ell_b^2\, W_{1122}(T) = \frac{\ell_{\mathrm{P}}^4}{\mathcal{N}} \sum_{j_1, j_2} \prod_e (2j_e + 1)\, \{6j\}\, e^{-\frac{\alpha}{2}\sum_i \delta j_i^2 + i\vartheta \sum_i (j_i + \frac{1}{2})} \times \\ [C^2(j_1) - C^2(j_0)]\, [C^2(j_2) - C^2(j_0)]. \tag{5.1}$$

Before proceeding, let us stress the logic: we introduced a flat boundary state, an equilateral tetrahedron with label j_0; this enters into the choice of the Gaussian as well as into the definition of the field insertions. Then, to have a semi–classical behaviour of its geometry, we consider the large j_0 regime. In this regime, (5.1) is approximated by

$$\ell_a^2\, \ell_b^2\, W_{1122}(T) \simeq \frac{\ell_{\mathrm{P}}^4}{\mathcal{N}} \sum_{j_1, j_2} \prod_e (2j_e + 1)\, \frac{4 j_0^2\, \delta j_1\, \delta j_2}{\sqrt{6\pi V(j_e)}} \times \\ \left(e^{iS_{\mathrm{R}}[j_e] + i\frac{\pi}{4}} + e^{-iS_{\mathrm{R}}[j_e] - i\frac{\pi}{4}}\right) e^{-\frac{\alpha}{2}\sum_i \delta j_i^2 + i\vartheta \sum_i (j_i + \frac{1}{2})}. \tag{5.2}$$

This is the expression we now evaluate analytically to study the asymptotic behaviour of the 2-point function, which is to be compared with (3.1). As in [8], the Gaussian implies $\delta j_i \ll 1$ in the sum, thus we can expand the Regge action around j_0,

$$S_{\mathrm{R}}[j_e] = \sum_e (j_e + \frac{1}{2})\theta_e(j_e) \simeq \sum_e (j_0 + \frac{1}{2})\,\vartheta + \sum_i \frac{\partial S_{\mathrm{R}}}{\partial j_i}|_{j_e = j_0} \delta j_i + \frac{1}{2}\sum_{i,k} \frac{\partial^2 S_{\mathrm{R}}}{\partial j_i \partial j_k}|_{j_e = j_0} \delta j_i \delta j_k. \tag{5.3}$$

Using (2.2), we have $\frac{\partial S_{\mathrm{R}}}{\partial j_i}|_{j_e = j_0} = \vartheta$. Introducing $G_{ik} = \frac{\partial \theta_i}{\partial j_k}|_{j_e = j_0}$, we can write

$$S_{\mathrm{R}}[j_e] \simeq 4\,\vartheta\,(j_0 + \frac{1}{2}) + \vartheta \sum_{i=1}^{2} (j_i + \frac{1}{2}) + \frac{1}{2}\sum_{i,k=1}^{2} G_{ik} \delta j_i \delta j_k. \tag{5.4}$$

G_{ik} can be calculated from elementary geometry; for the equilateral tetrahedron at hand, $G_{11} = G_{22} = -\frac{\sqrt{2}}{3 j_0}$, $G_{12} = G_{21} = -\frac{\sqrt{2}}{j_0}$. If we insert this expansion into (5.2), the two exponentials become

$$e^{i\frac{\pi}{4} + 4i\vartheta(j_0 + \frac{1}{2}) + 2i\vartheta \sum_i (j_i + \frac{1}{2}) + \frac{i}{2}\sum_{i,k} G_{ik}\delta j_i \delta j_k - \frac{\alpha}{2}\sum_i \delta j_i^2} + e^{-i\frac{\pi}{4} - 4i\vartheta(j_0 + \frac{1}{2}) - \frac{i}{2}\sum_{i,k} G_{ik}\delta j_i \delta j_k - \frac{\alpha}{2}\sum_i \delta j_i^2}.$$

The first one is a rapidly oscillating term in j_i, which vanishes when we perform the sum (5.2), so that we can consider only the second term. Furthermore, notice that at this order in the expansion in δj_i, we have $V(j_e) \sim V(j_0)$, thus the volume does not enter the sum (5.2); absorbing it in the normalisation, together with the phase $e^{-i\frac{\pi}{4} - 4i\vartheta(j_0 + \frac{1}{2})}$, we have

$$\ell_a^2\, \ell_b^2\, W_{1122}(T) \simeq \frac{1}{\mathcal{N}}\, 4\, j_0^2\, \ell_{\mathrm{P}}^4 \sum_{j_1, j_2} \prod_e (2j_e + 1)\, \delta j_1\, \delta j_2\, e^{-\frac{\alpha}{2}\sum_i \delta j_i^2 - \frac{i}{2}\sum_{i,k} G_{ik}\delta j_i \delta j_k}. \tag{5.5}$$

The sum can be easily performed by approximating it with a Gaussian integral. In fact, in the limit $j \mapsto \infty$, $\ell_{\mathrm{P}} \mapsto 0$, we can write

$$\sum_{j_1, j_2} \prod_e (2j_e + 1) \sim (2j_0 + 1)^4 \int_0^\infty (2j_1 + 1) dj_1 \int_0^\infty (2j_2 + 1) dj_2 \simeq (2j_0)^6 \int dj_1 dj_2,$$

and also the factor $(2j_0)^6$ can be clearly absorbed in the normalisation. Changing variables from j_i to $z_i := \delta j_i$, we have $dj_i = dz_i$ and

$$\ell_a^2\,\ell_b^2\,W_{1122}(T) \simeq \frac{1}{\mathcal{N}}\,4\,j_0^2\,\ell_\mathrm{P}^4 \int dz_1 dz_2\; z_1 z_2\; e^{-\frac{1}{2}z_i A_{ij} z_j}, \tag{5.6}$$

where $A_{ij} = \alpha\delta_{ij} - iG_{ij}$. At this point, we can fix the value of α to tune the correct vacuum boundary state. If we choose $\alpha = \frac{4}{3j_0}$, which is consistent with (3.4), we have

$$A_{ij} = \alpha \begin{pmatrix} 1 + i\cot\theta & -\frac{i}{\sin\theta} \\ -\frac{i}{\sin\theta} & 1 + i\cot\theta \end{pmatrix}. \tag{5.7}$$

This can be easily recognised to be the kernel of an harmonic oscillator (see the appendix), with frequency

$$\omega = \frac{\theta}{T} \simeq \frac{8}{3}\frac{1}{j_0\ell_\mathrm{P}} \equiv \frac{2\alpha}{\ell_\mathrm{P}}. \tag{5.8}$$

Therefore, the vacuum boundary state together with the PR amplitude reproduce in the semi–classical limit the kernel of an harmonic oscillator; using the boundary geometry to evaluate $\ell_a^2\,\ell_b^2 = (j_0\ell_\mathrm{P})^4$,

$$W_{1122}(T) \simeq \frac{4}{j_0^2}\,A_{12}^{-1} = \frac{4}{j_0^2}\frac{e^{i\theta}}{2\alpha} = \frac{8}{j_0^2\ell_\mathrm{P}}\frac{e^{i\omega T}}{2\omega}. \tag{5.9}$$

The expression (5.9) picks contributions from a single frequency. However, suppose now that our tetrahedron is part of a discretisation of spacetime. Then we expect to be able to sum over all frequencies admitted by the lattice, and (5.9) becomes

$$W_{1122}(T) = \frac{8}{j_0^2\ell_\mathrm{P}} \sum_p \frac{e^{i\omega_p T}}{2\omega_p}, \tag{5.10}$$

which coincides with (3.1), once we set $f_\lambda = 8$, and we make the usual lattice approximation $\int d^2p \sim \frac{1}{(j_0\ell_\mathrm{P})^2}\sum_p$ (where $j_0\ell_\mathrm{P}$ provides the characteristic length of the boundary geometry).

6. Dependence on the distance

To study the dependence on the spacetime distance ℓ of the 2-point function, it is convenient to consider its absolute value. From (5.9), we read that at large scales

$$W(\ell) := |W_{1122}(\ell_\mathrm{P} j_0)| = \frac{3}{2\ell}, \tag{6.1}$$

thus reproducing the expected inverse power law of the 3d free graviton.

However, we can now use the exact expression (5.1) to study the dependence on the distance at short scales. Notice that corrections to the inverse power law are expected to arise at short scales; in the context of conventional QFT, these are understood as graviton self–energies (see for instance [16]). In our approach, they are indeed present, and

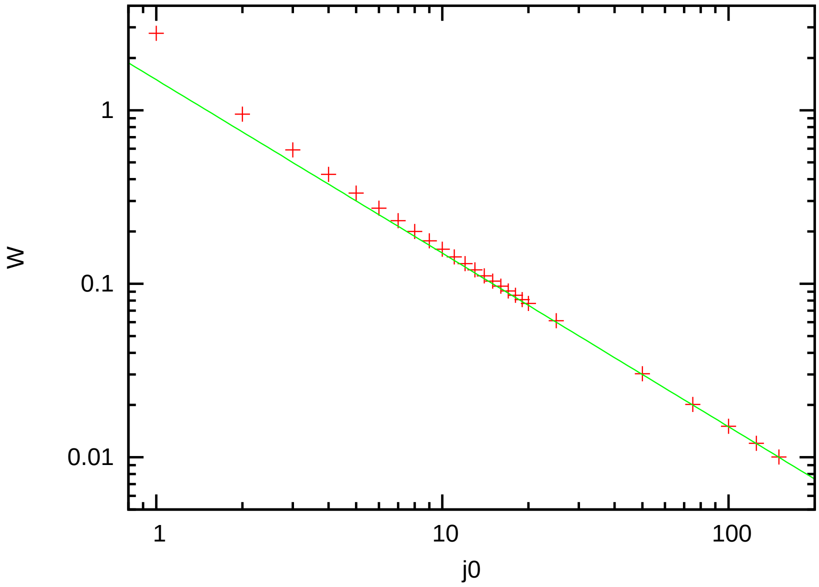

Figure 2: The crosses are the numerical evaluations of (5.1), plotted against the asymptotic behaviour $\frac{3}{2j_0}$ on a bi-logarithmic scale. The agreement is very good for $j_0 \geq 50$, while deviations appear for smaller values.

come from higher order terms in the expansion (5.3) as well as from departures from the asymptotic (2.4). Here, we do not attempt an analytical evaluation of these corrections, but simply briefly report a numerical study: in Fig.2 we plot some numerical values of (5.1) when varying j_0. We see that these are in very good agreement with the asymptotic behaviour (6.1) for $j_0 \geq 50$. For smaller values of j_0, we begin to see deviations from the inverse power law. These deviations can be fitted by a series of inverse powers, which are the small scale quantum corrections. For instance, allowing powers down to -4, we obtain the following fitting function,

$$W(\ell) \simeq \frac{3}{2\ell}[1 + \ell_{\mathrm{P}}\frac{0.67}{\ell} - \ell_{\mathrm{P}}^2\frac{0.74}{\ell^2} + \ell_{\mathrm{P}}^3\frac{0.92}{\ell^3}]. \tag{6.2}$$

It should be added that these corrections are gauge–dependent quantities, just as the free propagator. In the conventional perturbative expansion of 3d GR, the self–energies vanish in the gauge in which the free propagator vanishes, so that the full propagator is a pure gauge. At the present stage of investigation, there is no reason to think that the spinfoam formalism should modify this picture, thus we expect these corrections to be again pure gauge quantities; namely, we expect them to vanish in the same gauge in which the free term vanishes.

7. Conclusions

We considered the recently appeared proposal for the extraction of the 2-point function of linearised quantum gravity from the spinfoam formalism. To clarify some of the geometrical issues, we focused on a 3d toy model where spacetime is discretised by a single

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tetrahedron. We showed that, upon properly fixing the width of the Gaussian used as the vacuum boundary state, the kernel for the space perturbations around flat space behaves as the kernel of an harmonic oscillator, with frequency given by $\frac{8}{3\ell}$, where $\ell = j_0 \ell_{\rm P}$ is the characteristic length of the boundary geometry. Therefore, the propagator can be reconstructed as a collection of harmonic oscillators, yielding the expected inverse power law in the large scale limit. Furthermore, we numerically studied the short scale behaviour, hinting that the $1/\ell$ behaviour is corrected by a series of higher inverse powers. The proposal not only reproduces the expected large scale behaviour, but it is also useful to study the short scale quantum corrections.

The main open issue is the extension of these results to many tetrahedra. On the one hand, this is an important check of the stability of the result; on the other hand, it would allow the study of the other components of the propagator, and thus recover its full tensorial structure. In particular, notice that the full tensorial structure is needed to prove in a more consistent way that the 2-point function computed here is just a pure gauge.

We hope that the clarifications here presented will help the calculations in the 4d case.

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A. 2-point function from the kernel for the harmonic oscillator

Consider an harmonic oscillator, with mass $m = 1$, in units $\hbar = 1$, and let us compute the 2-point function $G(t_1, t_2) = \langle 0|x(t_1)x(t_2)|0\rangle$. This can be done in the canonical formalism, giving

$$\langle 0|x(t_1)x(t_2)|0\rangle = \langle 0|x e^{iH(t_2-t_1)} x|0\rangle = \frac{1}{2\omega} e^{i\frac{3}{2}\omega T}, \tag{A.1}$$

with $t_2 - t_1 := T$. However, it can also be computed starting from the propagation kernel $W[x_1, x_2, T]$, as in (1.1). We have

$$G(t_1, t_2) = \frac{1}{\mathcal{N}} \int dx_1\, dx_2\, W[x_1, x_2, T]\, \Psi_0[x_1]\, x_1\, \Psi_0[x_2]\, x_2, \tag{A.2}$$

where the normalisation is $\mathcal{N} = \int dx_1\, dx_2\, W[x_1, x_2, T]\, \Psi_0[x_1]\, \Psi_0[x_2]$, and $\Psi_0[x]$ is the vacuum state. Using the well known expressions (see for instance [1])

$$\Psi_0[x] = \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{1}{2}\frac{\omega}{\hbar}x^2}, \qquad W[x_1, x_2, T] = \sqrt{\frac{\omega}{2\pi i \sin\omega T}} e^{-i\frac{\omega}{2}\frac{(x_1^2+x_2^2)\cos\omega T - 2x_1x_2}{\sin\omega T}}, \tag{A.3}$$

we obtain

$$G(t_1, t_2) = \frac{1}{\mathcal{N}} \int dx_1\, dx_2\, x_1\, x_2\, e^{-\frac{1}{2}x_i A_{ij} x_j}, \tag{A.4}$$

with

$$A_{ij} = \omega \begin{pmatrix} 1 + i\cot\omega T & -\frac{i}{\sin\omega T} \\ \\ -\frac{i}{\sin\omega T} & 1 + i\cot\omega T \end{pmatrix}. \tag{A.5}$$

The expression (A.4) is a Gaussian integral that can be readily evaluated, giving

$$G(t_1, t_2) = A^{-1}_{12} = \frac{1}{2\omega} e^{i\omega T}. \tag{A.6}$$

Up to the vacuum energy contribution $e^{i\frac{1}{2}\omega T}$, the result coincides with the canonical evaluation (A.1).

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